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On the non-linear interaction of inertial modes

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It is shown that the principal non-linear interaction of inviscid inertial modes does not produce a resonant response in the steady geostrophic circulation. The rectified geostrophic flow manifested in a closed rotating container by a periodic excitation seems to result from a combination of viscous and non-linear effects within the boundary layers.

1. Introduction

A number of rotating fluid experiments have been conducted (e.g. Fultz 1959; Malkus 1968) which either produce or require pure inertial modes in closed containers. In each of these, as the level of excitation is increased beyond the linear range, a steady zonal flow invariably appears as a manifestation of some non-linear interaction. It has been conjectured that this interaction takes place within the viscous boundary layer at the containing wall. Indeed, Busse (1968) has confirmed this in one particular case and has shown that the most important regions in this respect are those surrounding the critical latitudes (where the basic oscillation frequency is exactly twice the component of the rotation vector along the normal to the boundary).

The object of this note is to present further evidence in support of this contention by proving that a purely inviscid, non-linear interaction of inertial modes does not produce a 'significant' rectified motion in a closed configuration. Thus, it appears that a combination of viscous and non-linear effects within the boundary layer is the essential mechanism in the development of the steady interior circulations observed.

2. Formulation

If the scales of length, time and relative velocity (in the uniformly rotating frame for which $\mathbf{\Omega} = \Omega\hat{\mathbf{k}}$) are characterized by L, Ω^{-1} and $\epsilon\Omega L$ where ϵ is small, then the dimensionless inviscid boundary value problem is

$$\frac{\partial}{\partial t}\mathbf{q}+2\hat{\mathbf{k}}\times\mathbf{q} = -\nabla p-\epsilon\{\tfrac{1}{2}\nabla(\mathbf{q}\cdot\mathbf{q})+(\nabla\times\mathbf{q})\times\mathbf{q}\}, \tag{1}$$

$$\nabla\cdot\mathbf{q} = 0, \tag{2}$$

with

$$\mathbf{q}\cdot\mathbf{n} = 0 \tag{3}$$

on Σ, the surface of the container whose normal is $\mathbf{n}$. Let the top and bottom surfaces of the container, $\Sigma = \Sigma_T + \Sigma_B$, be represented by

$$z = f(x, y)\,(\Sigma_T), \quad z = -g(x, y)\,(\Sigma_B), \tag{4}$$

so that

$$\left.\begin{aligned} \mathbf{n}_T &= \hat{\mathbf{k}} - \nabla f = (1 + (\nabla f)^2)^{\frac{1}{2}}\,\hat{\mathbf{n}}_T, \\ \mathbf{n}_B &= -\hat{\mathbf{k}} - \nabla g = (1 + (\nabla g)^2)^{\frac{1}{2}}\,\hat{\mathbf{n}}_B. \end{aligned}\right\} \tag{5}$$

We consider container configurations for which the general solution of the inviscid *linear* problem is a superposition of normal modes, of which there are two types. There are an infinite number of inertial oscillations typically represented by

$$\mathbf{q} = \mathbf{Q}_m(\mathbf{r}) \exp(i\lambda_m t), \quad p = \Phi_m(\mathbf{r}) \exp(i\lambda_m t), \tag{6}$$

and a single geostrophic mode, a steady zonal flow, of the form

$$\mathbf{q} = \mathbf{q}(\mathbf{r}), \quad p = \phi(\mathbf{r}). \tag{7}$$

The equations for the modes are obtaining by setting $\epsilon = 0$ in (1) and substituting the particular functional forms for velocity and pressure. For example,

$$i\lambda_m \mathbf{Q}_m + 2\hat{\mathbf{k}} \times \mathbf{Q}_m = -\nabla \Phi_m, \quad \nabla \cdot \mathbf{Q}_m = 0, \tag{8}$$

with

$$\mathbf{Q}_m \cdot \hat{\mathbf{n}} = 0 \quad \text{on} \quad \Sigma.$$

These modes have the following important properties (see Greenspan 1968):

$$\left.\begin{aligned} &\text{(i)} \quad \lambda_m \text{ is real and } |\lambda_m| < 2; \\ &\text{(ii)} \quad \int \mathbf{Q}_n \cdot \mathbf{Q}_m^{\dagger}\, dV = 0, \end{aligned}\right\} \tag{9}$$

for $n \neq m$, where V is the volume of the container and $\dagger$ denotes complex conjugate;

$$\text{(iii)} \quad \phi(\mathbf{r}) = \phi(h), \tag{10}$$

where $h = f + g$;

$$\text{(iv)} \quad \mathbf{q}(\mathbf{r}) = \mathbf{q}(x, y) = -\frac{1}{2}\frac{\partial \phi(h)}{\partial h}\, \mathbf{n}_T \times \mathbf{n}_B; \tag{11}$$

$$\text{(v)} \quad \oint_{\mathfrak{C}} \langle(\mathbf{Q}_m)\rangle \cdot \mathbf{ds} = 0, \tag{12}$$

where $\mathfrak{C}$ is a closed geostrophic contour of constant height, $h = f + g$ (whose tangent vector is $\mathbf{n}_T \times \mathbf{n}_B / |\mathbf{n}_T \times \mathbf{n}_B|$) and

$$\langle \psi \rangle = \int_{-g}^{f} \psi(x, y, z)\, dz. \tag{13}$$

The general, linear, inviscid solution is assumed to be a synthesis of all the natural modes:

$$\mathbf{q}(\mathbf{r}, t) = \mathbf{q}(x, y) + \sum_m a_m \mathbf{Q}_m(\mathbf{r}) \exp(i\lambda_m t). \tag{14}$$

Moreover, if $\mathbf{q}(\mathbf{r}, 0) = \mathbf{q}_*(\mathbf{r})$ at $t = 0$, then it follows from properties (ii) and (v) that

$$a_m = \frac{\int \mathbf{q}_* \cdot \mathbf{Q}_m^{\dagger}\, dV}{\int \mathbf{Q}_m \cdot \mathbf{Q}_m^{\dagger}\, dV}, \tag{15}$$

and
$$\mathbf{q} = \frac{1}{J(h)}\left[\oint_{\mathfrak{C}} \langle \mathbf{q}_* (\mathbf{r})\rangle \, . \, \mathbf{ds}\right] \mathbf{n}_T \times \mathbf{n}_B, \tag{16}$$
where
$$J(h) = h \oint_{\mathfrak{C}} |\, \mathbf{n}_T \times \mathbf{n}_B \,|\, ds.$$

We proceed now to construct a general solution of the equations of motion (1), (2) with the restriction that the non-linear interactions are weak, i.e. $\epsilon \ll 1$. To this end, the form of the linear solution, (14), is modified to account for a slow time variation on a scale dictated by the parameter ϵ^{-1}. Therefore, let

$$\left.\begin{aligned} \mathbf{q} &= \mathbf{q}(x,y,t) + \sum_m [A_m(t)\, \mathbf{Q}_m(\mathbf{r}) + A_m^\dagger(t)\, \mathbf{Q}_m^\dagger(\mathbf{r})], \\ p &= \phi(h,t) + \sum_m [A_m(t)\, \Phi_m(\mathbf{r}) + A_m^\dagger(t)\, \Phi_m^\dagger(\mathbf{r})], \end{aligned}\right\} \tag{17}$$

where $\mathbf{q}$ *is a solution of the steady geostrophic equations* and $\mathbf{Q}_m(\mathbf{r})$ are the modal functions identified earlier. This representation automatically satisfies the divergence equation, (2), and the boundary conditions. The explicit form of $\mathbf{q}$ (its dependence on t) and the coefficient functions $A_m(t)$ must now be determined.

3. Weak interactions

The substitution of (17) into (2) and use of the orthogonality and mean circulation conditions, (9) and (12), result in the following 'condensed' equations for the unknown functions:

$$\frac{dA_m}{dt} - i\lambda_m A_m = -\epsilon \int \mathbf{Q}_m^\dagger \, . \, (\nabla \times \mathbf{q}) \times \mathbf{q}\, dV, \tag{18}$$

$$h \oint_{\mathfrak{C}} \frac{\partial \mathbf{q}}{\partial t} \, . \, \mathbf{ds} = -\epsilon \oint_{\mathfrak{C}} \langle (\nabla \times \mathbf{q}) \times \mathbf{q} \rangle \, . \, \mathbf{ds}. \tag{19}$$

The integrals on the right-hand sides are functions of all the unknown coefficients (see the appendix for the explicit forms) and at this point the assumption of weak non-linearity, or small ϵ, must be invoked. This is done through the introduction of a perturbation expansion whose structure also incorporates multiple time scales in order to maintain a uniformly valid solution:

$$\left.\begin{aligned} A_m(t) &= A_{m0}(t,\tau,\mathrm{t},\ldots) + \epsilon A_{m1}(t,\tau,\mathrm{t},\ldots) + \ldots, \\ \mathbf{q}(x,y,t) &= \mathbf{q}_0(x,y,t,\tau,\mathrm{t},\ldots) + \epsilon \mathbf{q}_1(x,y,t,\tau,\mathrm{t},\ldots) + \ldots, \\ \phi(h,t) &= \phi_0(h,t,\tau,\mathrm{t},\ldots) + \epsilon \phi_1(h,t,\tau,\mathrm{t},\ldots) + \ldots, \end{aligned}\right\} \tag{20}$$

where
$$\tau = \epsilon t, \quad \mathrm{t} = \epsilon^2 t, \ldots. \tag{21}$$

The time derivatives then appear as follows:

$$\left.\begin{aligned} \frac{dA_m}{dt} &= \frac{\partial A_{m0}}{\partial t} + \epsilon\left(\frac{\partial A_{m1}}{\partial t} + \frac{\partial A_{m0}}{\partial \tau}\right) + \ldots, \\ \frac{\partial}{\partial t}\mathbf{q} &= \frac{\partial}{\partial t}\mathbf{q}_0 + \epsilon\left(\frac{\partial}{\partial t}\mathbf{q}_1 + \frac{\partial}{\partial \tau}\mathbf{q}_0\right) + \ldots. \end{aligned}\right\} \tag{22}$$

Upon substituting these series in (18) and (19) and equating powers of ϵ, we find that to lowest order

$$A_{m0} = a_{m0}(\tau, \mathrm{t}, \ldots) \exp(i\lambda_m t), \tag{23}$$

and

$$\frac{\partial}{\partial t} \mathbf{q}_0 = 0,$$

or

$$\mathbf{q}_0 = \mathbf{q}_0(x, y, \tau, \mathrm{t}, \ldots) = -\frac{1}{2}\left(\frac{\partial}{\partial h}\phi_0(h, \tau, \mathrm{t}, \ldots)\right) \mathbf{n}_T \times \mathbf{n}_B. \tag{24}$$

(The essential details for all the calculations in this section are presented in the appendix.) At the next step, that is terms of order ϵ, it becomes necessary to eliminate secular terms (e.g. $t \exp(i\lambda_m t)$) to assure a uniformly valid expansion. The conditions for this imply that

$$A_{m0} = \alpha_{m0}(\mathrm{t}) \exp\left\{\tau\left[\int \mathbf{Q}_m^\dagger \cdot (\mathbf{q}_0 \times \nabla \times \mathbf{Q}_m + \mathbf{Q}_m \times \nabla \times \mathbf{q}_0)\, dV\right] + i\lambda_m t\right\}, \tag{25}$$

and

$$h \oint_{\mathfrak{C}} \frac{\partial \mathbf{q}_0}{\partial \tau} \cdot \mathbf{ds} = \sum_m |A_{m0}|^2 \oint_{\mathfrak{C}} \langle \mathbf{Q}_m^\dagger \times \nabla \times \mathbf{Q}_m + \mathbf{Q}_m \times \nabla \times \mathbf{Q}_m^\dagger \rangle \cdot \mathbf{ds}. \tag{26}$$

These equations can be greatly simplified by manipulating the integrals involving $\mathbf{Q}_m$. For example, it is a straightforward calculation, using (8), to show that

$$\mathbf{Q}_m^\dagger \times \nabla \times \mathbf{Q}_m + \mathbf{Q}_m \times \nabla \times \mathbf{Q}_m^\dagger = \frac{1}{i\lambda_m} \frac{\partial}{\partial z}(\mathbf{Q}_m^\dagger \times \mathbf{Q}_m),$$

or equivalently,

$$\langle \mathbf{Q}_m^\dagger \times \nabla \times \mathbf{Q}_m + \mathbf{Q}_m \times \nabla \times \mathbf{Q}_m^\dagger \rangle = \frac{1}{i\lambda_m} (\mathbf{Q}_m^\dagger \times \mathbf{Q}_m)]_B^T, \tag{27}$$

where the bracket $]_B^T$ indicates evaluations at the top and bottom boundaries of the container. Furthermore,

$$(\mathbf{n}_T \times \mathbf{n}_B) \cdot (\mathbf{Q}_m^\dagger \times \mathbf{Q}_m) = (\mathbf{n}_T \cdot \mathbf{Q}_m^\dagger)(\mathbf{n}_B \cdot \mathbf{Q}_m) - (\mathbf{n}_T \cdot \mathbf{Q}_m)(\mathbf{n}_B \cdot \mathbf{Q}_m^\dagger), \tag{28}$$

which according to (8) is identically zero at Σ_T and Σ_B. This result, together with the identity

$$\mathbf{ds} = \frac{\mathbf{n}_T \times \mathbf{n}_B}{|\mathbf{n}_B \times \mathbf{n}_B|} ds,$$

allows the reduction of (26) to

$$h \oint_{\mathfrak{C}} \frac{\partial \mathbf{q}_0}{\partial \tau} \cdot \mathbf{ds} = 0, \tag{29}$$

or more simply, just

$$\frac{\partial \phi_0}{\partial \tau} = 0. \tag{30}$$

(According to (11),

$$h \oint_{\mathfrak{C}} \frac{\partial \mathbf{q}_0}{\partial \tau} \cdot \mathbf{ds} = -\frac{h}{2}\left[\frac{\partial}{\partial \tau}\phi_0(h, \tau)\right] \oint_{\mathfrak{C}} |\mathbf{n}_T \times \mathbf{n}_B| \cdot \mathbf{ds}.$$

Equation (29) is then equivalent to

$$\frac{\partial \phi_0}{\partial \tau} = 0,$$

which proves the statement (30).)

Thus, we conclude that there are no resonant non-linear interactions of inertial oscillations which produce an $O(1)$ effect on the steady geostrophic mode in the long time scale τ. Minor $O(\epsilon)$ oscillations along constant-height contours and equally small non-geostrophic circulations are generated, but no major resonant rectification occurs.

The main effect, in this period of time, of a non-linear interaction on an inertial mode is to shift the frequency of oscillation, but the amplitude remains unchanged. This can be established from (25) by computing the absolute value of the modal coefficient:

$$|A_{m0}|^2 = |\alpha_{m0}|^2 \exp\left\{\tau \int [\mathbf{Q}_m^\dagger . (\mathbf{q}_0 \times \nabla \times \mathbf{Q}_m) + \mathbf{Q}_m . (\mathbf{q}_0 \times \nabla \times \mathbf{Q}_m^\dagger)] \, dV\right\}. \quad (31)$$

The volume integral in this expression is, however, identically zero, a fact established in the following way. Note that

$$\int [\mathbf{Q}_m^\dagger . (\mathbf{q}_0 \times \nabla \times \mathbf{Q}_m) + \mathbf{Q}_m . (\mathbf{q}_0 \times \nabla \times \mathbf{Q}_m^\dagger)] \, dV$$

$$= -\int \mathbf{q}_0 . [\mathbf{Q}_m^\dagger \times \nabla \times \mathbf{Q}_m + \mathbf{Q}_m \times \nabla \times \mathbf{Q}_m^\dagger] \, dV,$$

or, using (27),

$$= -\frac{1}{i\lambda_m} \int \mathbf{q}_0 . \mathbf{Q}_m^\dagger \times \mathbf{Q}_m]_B^T \, dx \, dy.$$

However, the last integrand is proportional to the quantity

$$(\mathbf{n}_T \times \mathbf{n}_B) . \mathbf{Q}_m^\dagger \times \mathbf{Q}_m]_B^T,$$

which is zero according to (28) and the boundary condition, $\mathbf{n} . \mathbf{Q}_m = 0$ on Σ. Thus, (31) becomes

$$|A_{m0}|^2 = |\alpha_{m0}|^2, \quad \text{or} \quad \frac{\partial}{\partial \tau} |A_{m0}|^2 = 0,$$

proving that the amplitude is a constant to this degree of approximation, within the time scale indicated. The effects of possible resonant triads, when the frequencies of two modes add exactly to that of a third have not been considered here except, of course, for the case of steady motion (zero frequency). The more general resonant-triad interaction cannot have an important influence on the primary geostrophic mode until $t \sim \epsilon^{-2}$.

Longuet-Higgins & Gill (1967) examined the properties of resonant-triad interactions of plane, planetary waves in a study based on the β-plane approximation of the potential vorticity equation. The main interactions and energy transfer were shown to occur in the vicinity of a particular pair of latitude circles. In addition, it was found that zonal currents cannot gain or lose any energy by the mechanism of non-linear wave interactions, a conclusion which is, of course, a special case of the general result given here.

4. Conclusion

It has been shown that a general, inviscid, non-linear interaction of discrete inertial modes does not produce a significant (i.e. $O(1)$) steady circulation in a closed container in an $O(\epsilon^{-1})$ interval of time.

The principal cause of the steady currents which do develop from oscillatory disturbances or excitations seems to be non-linear processes within the viscous boundary layers. This mechanism is also responsible for any significant changes in the amplitudes of contained inertial modes in the $O(\epsilon^{-1})$ time period. Large changes might occur for $t \sim \epsilon^{-2}$, but to examine this possibility, it is necessary to extend the asymptotic analysis to still higher order.

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Appendix

The relevant details of the analysis summarized in §2 are set forth here.

The expanded versions of (18) and (19) are

$$\begin{aligned}\frac{d}{dt}A_m - i\lambda_m A_m = -\epsilon\Bigg\{&\int \mathbf{Q}_m^\dagger \cdot (\nabla\times\mathbf{q})\times\mathbf{q}\,dV \\ &+\sum_n A_n\int[(\nabla\times\mathbf{q})\times\mathbf{Q}_n\cdot\mathbf{Q}_m^\dagger - \mathbf{q}\times\nabla\times\mathbf{Q}_n\cdot\mathbf{Q}_m^\dagger]\,dV \\ &+\sum_n A_n^\dagger\int[(\nabla\times\mathbf{q})\times\mathbf{Q}_n^\dagger\cdot\mathbf{Q}_m^\dagger - \mathbf{q}\times\nabla\times\mathbf{Q}_n^\dagger\cdot\mathbf{Q}_m^\dagger]\,dV \\ &+\sum_{n,l}\Bigg[A_nA_l\int\mathbf{Q}_m^\dagger\cdot(\nabla\times\mathbf{Q}_n)\times\mathbf{Q}_l\,dV \\ &+A_nA_l^\dagger\int\mathbf{Q}_m^\dagger\cdot(\nabla\times\mathbf{Q}_n)\times\mathbf{Q}_l^\dagger\,dV \\ &+A_n^\dagger A_l\int\mathbf{Q}_m^\dagger\cdot(\nabla\times\mathbf{Q}_n^\dagger)\times\mathbf{Q}_l\,dV \\ &+A_n^\dagger A_l^\dagger\int\mathbf{Q}_m^\dagger\cdot(\nabla\times\mathbf{Q}_n^\dagger)\times\mathbf{Q}_l^\dagger\,dV\Bigg]\Bigg\},\end{aligned} \tag{A1}$$

$$\begin{aligned}h\oint_{\mathfrak{C}}\frac{\partial}{\partial t}\mathbf{q}\cdot\mathbf{ds} = -\epsilon\Bigg\{&\sum_n A_n\oint_{\mathfrak{C}}(\nabla\times\mathbf{q})\times\langle\mathbf{Q}_n\rangle\cdot\mathbf{ds} \\ &+\sum_n A_n^\dagger\oint_{\mathfrak{C}}(\nabla\times\mathbf{q})\times\langle\mathbf{Q}_n^\dagger\rangle\cdot\mathbf{ds} \\ &+\sum_{n,l}\Bigg[A_nA_l\oint_{\mathfrak{C}}\langle(\nabla\times\mathbf{Q}_n)\times\mathbf{Q}_l\rangle\cdot\mathbf{ds} \\ &+A_nA_l^\dagger\oint_{\mathfrak{C}}\langle(\nabla\times\mathbf{Q}_n)\times\mathbf{Q}_l^\dagger\rangle\cdot\mathbf{ds} \\ &+A_n^\dagger A_l\oint_{\mathfrak{C}}\langle(\nabla\times\mathbf{Q}_n^\dagger)\times\mathbf{Q}_l\rangle\cdot\mathbf{ds} \\ &+A_n^\dagger A_l^\dagger\oint_{\mathfrak{C}}\langle(\nabla\times\mathbf{Q}_n^\dagger)\times\mathbf{Q}_l^\dagger\rangle\cdot\mathbf{ds}\Bigg]\Bigg\}.\end{aligned} \tag{A2}$$

If, now, the expansions given in (20) are substituted into the foregoing and the coefficients of the same powers of ϵ equated, then an infinite set of equations is obtained. Those that correspond to ϵ^0 and ϵ^1 are

$$\frac{\partial A_{m0}}{\partial t} - i\lambda_m A_{m0} = 0, \tag{A 3}$$

$$\begin{aligned}
\frac{\partial A_{ml}}{\partial t} - i\lambda_m A_{ml} + \frac{\partial A_{m0}}{\partial \tau} = {} & \int \mathbf{Q}_m^\dagger \,.\, \mathbf{q}_0 \times \nabla \times \mathbf{q}_0 \, dV \\
& + \sum_n A_{n0} \int \mathbf{Q}_m^\dagger \,.\, [\mathbf{q}_0 \times \nabla \times \mathbf{Q}_n + \mathbf{Q}_n \times \nabla \times \mathbf{q}_0] \, dV \\
& + \sum_n A_{n0}^\dagger \int \mathbf{Q}_m^\dagger \,.\, [\mathbf{q}_0 \times \nabla \times \mathbf{Q}_n^\dagger + \mathbf{Q}_u \times \nabla \times \mathbf{q}_0] \, dV \\
& + \sum_{n,l} \bigg[A_{n0} A_{l0} \int \mathbf{Q}_m^\dagger \,.\, \mathbf{Q}_l \times \nabla \times \mathbf{Q}_n \, dV \\
& + A_{n0} A_{l0}^\dagger \int \mathbf{Q}_m^\dagger \,.\, \mathbf{Q}_l^\dagger \times \nabla \times \mathbf{Q}_n \, dV \\
& + A_{n0}^\dagger A_{l0} \int \mathbf{Q}_m^\dagger \,.\, \mathbf{Q}_l \times \nabla \times \mathbf{Q}_n^\dagger \, dV \\
& + A_{n0}^\dagger A_{l0}^\dagger \int \mathbf{Q}_m^\dagger \,.\, \mathbf{Q}_l^\dagger \times \nabla \times \mathbf{Q}_n^\dagger \, dV \bigg],
\end{aligned} \tag{A 4}$$

$$h \oint_{\mathfrak{C}} \frac{\partial \mathbf{q}_0}{\partial t} \,.\, d\mathbf{s} = 0, \tag{A 5}$$

$$\begin{aligned}
h \oint_{\mathfrak{C}} \left(\frac{\partial \mathbf{q}_1}{\partial t} + \frac{\partial \mathbf{q}_0}{\partial \tau} \right) . \, d\mathbf{s} = {} & \sum_n \bigg[A_{n0} \oint_{\mathfrak{C}} \langle \mathbf{Q}_n \rangle \times (\nabla \times \mathbf{q}_0) \,.\, d\mathbf{s} \\
& + A_{n0}^\dagger \oint_{\mathfrak{C}} \langle \mathbf{Q}_n^\dagger \rangle \times (\nabla \times \mathbf{q}_0) \,.\, d\mathbf{s} \bigg] \\
& + \sum_{n,m} \bigg[A_{n0} A_{m0} \oint_{\mathfrak{C}} \langle \mathbf{Q}_m \times \nabla \times \mathbf{Q}_n \rangle \,.\, d\mathbf{s} \\
& + A_{n0} A_{m0}^\dagger \oint_{\mathfrak{C}} \langle \mathbf{Q}_m^\dagger \times \nabla \times \mathbf{Q}_n \rangle \,.\, d\mathbf{s} \\
& + A_{n0}^\dagger A_{m0} \oint_{\mathfrak{C}} \langle \mathbf{Q}_m \times \nabla \times \mathbf{Q}_n^\dagger \rangle \,.\, d\mathbf{s} \\
& + A_{n0}^\dagger A_{m0}^\dagger \oint_{\mathfrak{C}} \langle \mathbf{Q}_m^\dagger \times \nabla \times \mathbf{Q}_n^\dagger \rangle \,.\, d\mathbf{s} \bigg].
\end{aligned} \tag{A 6}$$

Equations (A 3) and (A 5) indicate that $\mathbf{q}_0$ is steady, i.e. independent of the 'fast' variable t and that A_{m0} is purely oscillatory on the same time scale. The solutions for A_{m1} and $\mathbf{q}_1$ would contain terms proportional to t if such secular quantities are not suppressed. This may be done by choosing $\partial A_{m0}/\partial \tau$ and $\partial \mathbf{q}_0/\partial \tau$ to eliminate the inhomogeneous terms in (A 4) and (A 6) that also correspond to solutions of the respective homogeneous equations. We are led in this manner to (25) and (26) which formed the cornerstone of our analysis.

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